Recall:

Explicit formula for $\psi(x)$: For $2 \le T \le x$:

$$\psi(x) = x - \sum_{\rho : |\mathrm{Im}\,\rho| \le T} \frac{x^\rho}{\rho} + O\left(\frac{x(\log x)^2}{T}\right).$$

Zero-free region for $\zeta(s)$: $\exists$ constant $c > 0$ s.t. if $\rho = \sigma + it$ is a zero of $\zeta(s)$, then

$$\sigma \le 1 - \frac{c}{\log(2+|t|)}.$$

Theorem: There exists a constant $c_1 > 0$

$$\psi(x) = x + O\left(x e^{-c_1\sqrt{\log x}}\right).$$

Proof: We know that for $2 \le T \le x$, we have

$$\psi(x) = x - \sum_{|\mathrm{Im}\,\rho| \le T} \frac{x^\rho}{\rho} + O\left(\frac{x(\log x)^2}{T}\right).$$

For each ρ in the sum, we have $|x^\rho| = x^{\mathrm{Re}(\rho)} \le x^{1-\frac{c}{\log T}}$

Therefore $\psi(x) = x + O\left(x^{1-\frac{c}{\log T}} \sum_{|\mathrm{Im}\,\rho| \le T} \frac{1}{|\rho|}\right) + O\left(\frac{x(\log x)^2}{T}\right).$

There are $O(\log(1+n))$ zeros with $|\mathrm{Im}\,\rho| \in [n, n+1]$, therefore the sum is of size $O((\log T)^2)$.

Choose $T = \exp(\sqrt{\log x})$ to balance size of error terms. Then we find indeed that, for a suitable constant $c_1 > 0$

$$\psi(x) = x + O(x \exp(-c_1 \sqrt{\log x})). \qquad \square$$

$$\left(\text{say } c_1 = \min\left(\frac{c}{2}, \frac{1}{2}\right) \right).$$

Remark: Recall that for any $\varepsilon > 0$, $c > 0$, $N \in \mathbb{N}$,

$$(\log x)^N \ll \exp(c\sqrt{\log x}) \ll x^{\varepsilon}.$$

Corollary: (Prime number theorem)

$$\pi(x) = Li(x) + O(x \exp(-c_2 \sqrt{\log x})).$$

Proof: Recall $\theta(x) = \sum_{p \leq x} \log p$

We showed $\theta(x) = \psi(x) + O(\sqrt{x}(\log x)^2)$

$$\Longrightarrow \theta(x) = x + O(x \exp(-c_1 \sqrt{\log x})).$$

By partial summation: $\pi(x) = \dfrac{\theta(x)}{\log x} + \displaystyle\int_{3/2}^{x} \frac{\theta(t)}{t(\log t)^2}\, dt$

Hence $\pi(x) = \dfrac{x}{\log x} + \displaystyle\int_{3/2}^{x} \frac{1}{(\log t)^2}\, dt + O\left(\frac{x e^{-c_1\sqrt{\log x}}}{\log x} + \int_{3/2}^{x} \frac{e^{-c_1\sqrt{\log t}}}{(\log t)^2}\, dt \right)$

Main term: Note that $\left(\frac{1}{\log t}\right)' = -\frac{1}{t(\log t)^2}$.

Therefore $\frac{x}{\log x} + \int_{3/2}^{x} \frac{1}{(\log t)^2} dt =$

$$= \frac{x}{\log x} + \left[-\frac{t}{\log t}\right]_{3/2}^{x} + \int_{3/2}^{x} \frac{1}{\log t} dt = \operatorname{Li}(x) + O(1).$$

Error term: Note that $t \longmapsto t e^{-c_1\sqrt{\log t}}$

is an increasing function

Hence $\int_{3/2}^{x} \frac{e^{-c_1\sqrt{\log t}}}{(\log t)^2} dt \leq e^{-c_1\sqrt{\log x}} \cdot x \underbrace{\int_{3/2}^{x} \frac{1}{t(\log t)^2} dt}_{\text{this is } O(1)}$

$$\ll x \cdot e^{-c_1\sqrt{\log x}}.$$

$\boxtimes$

Our next goal is to study primes in arithmetic progressions ($a \bmod q$).

We already saw that if $(a, q) = 1$, then there are infinitely many primes $\equiv a \pmod q$. More precisely, if $\chi \pmod q$ is non-principal, then $L(1, \chi) \neq 0$ (Dirichlet's theorem) and this implies distribution results of the form

$$\sum_{\substack{p\le x\\ p\equiv a(q)}} \frac{\log p}{p} = \frac{1}{\varphi(q)}\log x + O_q(1)$$ (This is Mertens restricted to APs, and of course Mertens is weaker than PNT).

We want to obtain stronger results, an equivalent of prime number theorem in arithmetic progressions; but also with uniformity in q!
(From now on, consider q a parameter alongside x).

Denote $\pi(x;q,a) = \sum_{\substack{p\le x\\ p\equiv a(q)}} 1$,

$$\theta(x;q,a) = \sum_{\substack{p\le x\\ p\equiv a(q)}} \log p, \quad \psi(x;q,a) = \sum_{\substack{n\le x\\ n\equiv a(q)}} \Lambda(n).$$

For a character $\chi \pmod q$, define

$$\psi(x;\chi) := \sum_{n\le x} \chi(n)\Lambda(n).$$

Then $$\psi(x;q,a) = \frac{1}{\varphi(q)}\sum_{\chi} \overline{\chi}(a)\psi(x,\chi)$$

(by orthogonality of characters)

Also, we see that for $Re(s)>1$, we have
$$\sum_{n\geq 1} \frac{\chi(n)\Lambda(n)}{n^s} = -\frac{L'(s,\chi)}{L(s,\chi)}.$$

Hence, as in the proof of PNT we studied the properties of $\frac{\zeta'}{\zeta}(s)$, we now study $\frac{L'(s,\chi)}{L(s,\chi)}$, in particular the location of zeros ρ of $L(s,\chi)$ play an important role.

<u>Lemma</u>: Let χ a primitive character modulo q (for $q\geq 2$). Then for $-1\leq\sigma\leq 2$,
$$\frac{L'(\sigma+it)}{L(\sigma+it)} = \sum_{\rho:|s-\rho|\leq 1} \frac{1}{s-\rho} + O\left(\log\left(q(|t|+2)\right)\right).$$

(the sum is over non-trivial zeros ρ of $L(s,\chi)$ with multiplicity).

<u>Pf</u>: Same as for $\frac{\zeta'}{\zeta}(s)$, but there is no pole at $s=1$, see exercises.

Next we look at a variant of "zero-free region" for $L(s,\chi)$.

Theorem: There exists an absolute constant $c>0$ such that if χ a character mod q, then the region $R_q = \left\{ s : Re(s) > 1 - \frac{c}{\log(q(|Im(s)|+1))} \right\}$ contains no zero of $L(s,\chi)$, unless χ is quadratic, in which case $L(s,\chi)$ has at most one, necessarily real and simple zero in R_q.

Remark: Such a zero (if it exists), is called exceptional zero, or Siegel zero.

Proof: If $\chi \pmod q$ is induced by $\chi^* \pmod{q^*}$,

then $L(s,\chi) = L(s,\chi^*) \prod_{p|q} \left(1 - \frac{\chi^*(p)}{p^s}\right)$.

This implies zeros of $L(s,\chi)$ and $L(s,\chi^*)$ coincide in $Re(s)>0$, so we can assume WLOG χ is primitive. In particular, can also assume $\chi \neq \chi_0$ (since we know zero-free region for $\zeta(s)$).

Note that, for $\sigma > 1$, similarly as before,

$$\mathrm{Re}\left[-3\frac{L'}{L}(\sigma,\chi_0)-4\frac{L'}{L}(\sigma+it,\chi)-\frac{L'}{L}(\sigma+2it,\chi^2)\right]$$

$$=\sum_{(n,q)=1}\frac{\Lambda(n)}{n^\sigma}\left(3+4\,\mathrm{Re}(\chi(n)n^{-it})+\mathrm{Re}(\chi(n)^2n^{-2it})\right)$$

$(\chi(n)\,n^{-it}=e^{i\theta_n})$

$$=\sum_{(n,q)=1}\frac{\Lambda(n)}{n^\sigma}(3+4\cos\theta_n+\cos(2\theta_n))\geq 0.$$

We mimic the proof of zero-free region for $\zeta(s)$.
Let $1<\sigma\leq 2$.

$$-\frac{L'}{L}(\sigma,\chi_0)=\sum_{(n,q)=1}\frac{\Lambda(n)}{n^\sigma}\leq-\frac{\zeta'}{\zeta}(\sigma)=\frac{1}{\sigma-1}+O(1).$$

Let $\rho=\beta+it$ be a zero with $\beta>1-\frac{\delta}{\log(q(|t|+2))}$

$$\mathrm{Re}\left(-\frac{L'}{L}(\sigma+it,\chi)\right)\leq-\mathrm{Re}\left(\frac{1}{s-\rho}\right)+O(\log(q(|t|+2))).$$

$$=-\frac{1}{\sigma-\beta}+O(\log(q(|t|+2)))$$

<u>Case 1</u>: χ complex, so $\chi^2\neq\chi_0$.

The character χ^2 may not be primitive (so we cannot just apply previous lemma again).

Say χ^2 is induced from χ^* modulo $2\leq q^*\leq q$ (since $\chi^2\neq\chi_0$, this is crucial!!).

For $\operatorname{Re}(s) > 1$,

$$\frac{L'}{L}(s, \chi^2) - \frac{L'}{L}(s, \chi^*) = \sum_{p \mid q} \frac{d}{ds}\left(1 - \frac{\chi^*(p)}{p^s}\right)$$

$$= \sum_{p \mid q} \frac{\chi^*(p) p^{-s} \log p}{1 - \chi^*(p) p^{-s}} \ll \sum_{p \mid q} \log p \leq \log q$$

Hence $\operatorname{Re}\left(-\frac{L'}{L}(\sigma + 2it, \chi^2)\right) = -\operatorname{Re}\left(-\frac{L'}{L}(\sigma + 2it, \chi^*)\right)$

$+ O(\log q)$.

Using previous lemma, since χ^* primitive

$$\operatorname{Re}\left(-\frac{L'}{L}(\sigma + it, \chi^*)\right) \leq C \cdot \log(q(|t|+2)),$$

for some universal $C > 0$.

Therefore, putting it all together:

$$\frac{4}{\sigma - \beta} \leq \frac{3}{\sigma - 1} + O(\log(q(|t|+2))).$$

Choose $\sigma = 1 + \frac{4\delta}{\log(q(|t|+2))}$

Then $4 \frac{\log(q(|t|+2))}{5\delta} \leq 3 \frac{\log(q(|t|+2))}{4\delta} + O(\log(q(|t|+2)))$

Contradiction for δ small enough.

Case 2: χ quadratic, $\chi^2 = \chi_0$

Since $L(s,\chi_0) = \zeta(s) \prod_p \left(1 - \frac{1}{p^s}\right)$, then for

$\mathrm{Re}(s) > 1$, we have $\frac{L'(s,\chi_0)}{L(s,\chi_0)} - \frac{\zeta'}{\zeta}(s) \ll \log q$

(same argument as in previous case)

Hence $\mathrm{Re}\left(-\frac{L'}{L}(\sigma+2it, \chi^2)\right) = -\mathrm{Re}\left(\frac{\zeta'}{\zeta}(\sigma+2it)\right) + O(\log q)$

$\leq \mathrm{Re}\left(\frac{1}{\sigma-1+2it}\right) + O(\log(q(|t|+2)))$.

(using same arguments as before and partial fractional expansion for $\frac{\zeta'}{\zeta}(s)$, where the term $-\frac{1}{s-1}$ appears).

Case 2.1 $|t| \geq \frac{\delta}{\log q}$. Choose $\sigma = 1 + \frac{4\delta}{\log(q(|t|+2))}$

Then $\mathrm{Re}\left(\frac{1}{\sigma-1+2it}\right) = \frac{\sigma-1}{(\sigma-1)^2+(4t)^2} \ll \log q$.

So we have $\frac{4}{\sigma-\beta} \leq \frac{3}{\sigma-1} + O(\log(q(|t|+2)))$.

Obtain same contradiction as before ✓

Case 2.2 : $0 < |t| < \frac{\delta}{\log q}$.

Since χ is real, if ρ is a zero of $L(s,\chi)$, $\bar{\rho}$ is also a zero. Hence

$$-\frac{L'}{L}(\sigma,\chi) \le -\frac{1}{\sigma-\beta-it} - \frac{1}{\sigma-\beta+it} + O(\log q)$$

$$= -\frac{2(\sigma-\beta)}{(\sigma-\beta)^2+t^2} + O(\log q).$$

On the other hand,

$$-\frac{L'}{L}(\sigma,\chi) = \sum_n \frac{\Lambda(n)\chi(n)}{n^\sigma} \ge -\sum_n \frac{\Lambda(n)}{n^\sigma} = \frac{\zeta'}{\zeta}(\sigma)$$

$$\ge -\frac{1}{\sigma-1} + O(1).$$

$$\Longrightarrow -\frac{1}{\sigma-1} \le -\frac{2(\sigma-\beta)}{(\sigma-\beta)^2+t^2} + O(\log q).$$

Choose $\sigma = 1 + \frac{2\delta}{\log q}$. Then $|t| \le \frac{1}{2}(\sigma-1) \le \frac{1}{2}(\sigma-\beta)$

$$\Longrightarrow -\frac{\log q}{2\delta} \le -\frac{8/5}{(\sigma-\beta)} + O(\log q) \le -\frac{8/5\,\log q}{3\delta} + O(\log q)$$

(Recall $\beta > 1 - \frac{\delta}{\log(q(|t|+2))} \ge 1 - \frac{\delta}{\log q}$).

This is a contradiction if δ is small enough. ✓

Therefore $t=0$, so ρ must be real.

Uniqueness: Assume there are two zeros

$$1-\frac{\delta}{\log q} \le \beta_1 \le \beta_2 \le 1 \text{ (possibly equal)}.$$

Same argument shows $-\frac{1}{\sigma-1} \le -\frac{2\sigma-\beta_1-\beta_2}{(\sigma-\beta_1)(\sigma-\beta_2)} + O(\log q)$

Choose $\sigma = 1+\frac{4\delta}{\log q}$, we obtain the contradiction

$$-\frac{\log q}{4\delta} \le -\frac{8}{25}\frac{\log q}{\delta} + O(\log q).$$

$\square$

Lemma (Landau): Let $\chi_1 \pmod{q_1}$ and $\chi_2 \pmod{q_2}$ two distinct, real, primitive characters. Let β_j a real zero of $L(s,\chi_j)$. Then $\min(\beta_1,\beta_2) < 1-\frac{c_2}{\log q_1 q_2}$, for some (universal) $c_2 > 0$.

Proof: For $\sigma > 1$, we have

$$-\frac{\zeta'}{\zeta}(\sigma)-\frac{L'}{L}(\sigma,\chi_1)-\frac{L'}{L}(\sigma,\chi_2)-\frac{L'}{L}(\sigma,\chi_1\chi_2)$$

$$=\sum_n \frac{\Lambda(n)}{n^\sigma}\left(1+\chi_1(n)+\chi_2(n)+\chi_1\chi_2(n)\right)$$

$$=\sum_n \frac{\Lambda(n)}{n^\sigma}(1+\chi_1(n))(1+\chi_2(n))\geq 0.$$

Note that $\chi_1\chi_2$ is NOT the principal character modulo $q_1 q_2$ (otherwise χ_1 and χ_2 would induce some character modulo $q_1 q_2$).

So $$-\frac{\zeta'}{\zeta}(\sigma)=\frac{1}{\sigma-1}+O(1)$$

$$-\frac{L'}{L}(\sigma,\chi_i)\leq-\frac{1}{\sigma-\beta_i}+O(\log q_i),\quad i=1,2$$

$$-\frac{L'}{L}(\sigma,\chi_1\chi_2)\leq O(\log(q_1q_2))$$

So $$\frac{1}{\sigma-\beta_1}+\frac{1}{\sigma-\beta_2}\leq\frac{1}{\sigma-1}+O(\log q_1q_2).$$

Choose $\sigma=1+\dfrac{\delta}{\log(q_1q_2)}$

$$\Rightarrow \frac{2}{\sigma-\min(\beta_1,\beta_2)}\leq\frac{\log q_1q_2}{\delta}+O(\log q_1q_2)$$

$$\Rightarrow \min(\beta_1,\beta_2)\leq 1-\frac{1}{\log(q_1q_2)}\left(\frac{2}{\delta^{-1}+O(1)}-\delta\right). \square$$

Corollary: There is at most one character modulo q with a Siegel zero (a real zero with $\beta > 1 - \frac{c_3}{\log q}$).

Moreover, for $Q \geq 3$, there is at most one $q \leq Q$ for which it exists a primitive character χ mod q with a real zero $\beta > 1 - \frac{c_3}{\log Q}$.

Proof: Follows directly from previous lemma with $c_3 = \frac{c_2}{2}$. $\square$

Theorem: There exists a constant $c_4 > 0$ such that if $q \leq \exp(2c_4\sqrt{\log x})$ and $L(s,\chi)$ has no exceptional zero, then

$$\psi(x,\chi) = 1_{\chi=\chi_0}\, x + O(x\exp(-c_4\sqrt{\log x})).$$

If $L(s,\chi)$ has an exceptional zero β_1, then

$$\psi(x,\chi) = -\frac{x^{\beta_1}}{\beta_1} + O(x\exp(-c_4\sqrt{\log x})).$$

Proof: Exercise.

Remark: This shows that PNT in AP would follow easily if we have no Siegel zeros. We need a way to control size of Siegel zeros.

Also note that PNT in AP follows for modulus q bounded by constant (since there are finitely many characters of modulus q, their real zeros are uniformly bounded away from 1). So we now can prove for example

$$\pi(x;10,1) = \frac{1}{4} Li(x) + O\left(x \exp(-c\sqrt{\log x})\right).$$

There is a way to have some control of size of Siegel zero:

<u>Theorem</u> (Siegel) Let $\varepsilon > 0$. There exists a constant $C(\varepsilon) > 0$ such that for each real, primitive character χ (modulo q), we have

$$L(1,\chi) \geq C(\varepsilon) q^{-\varepsilon}.$$

<u>Remark</u>: Constant $C(\varepsilon)$ is ineffective.

<u>Remark</u>: As a corollary of Siegel's theorem, one can show that $\forall \varepsilon > 0$, $\exists\, C'(\varepsilon) > 0$ s.t. if β is a real zero of $L(s,\chi)$, then $\beta \leq 1 - C'(\varepsilon) q^{-\varepsilon}$.

If we assume this, one can prove a version of PNT in AP uniformly for all $q \leq (\log x)^A$, $\forall A > 0$.

Theorem (Siegel–Walfisz).
Let $A>0$. There exists $c=c(A)>0$ such that for all $q \le (\log x)^A$ and $(a,q)=1$,

$$\psi(x;q,a) = \frac{x}{\varphi(q)} + O\left(x e^{-c\sqrt{\log x}}\right)$$

and

$$\pi(x;q,a) = \frac{\mathrm{li}(x)}{\varphi(q)} + O\left(x e^{-c\sqrt{\log x}}\right).$$

It is possible to obtain stronger results if we are interested in average of error terms, rather than bounding each individual one.

Let $E(x,q) = \max\limits_{(a,q)=1} \left| \psi(x;q,a) - \frac{x}{\varphi(q)} \right|$.

Theorem (Bombieri–Vinogradov)
Let x, Q be such that $\frac{x^{1/2}}{(\log x)^A} \le Q \le x^{1/2}$, for some $A>0$.
Then $\sum\limits_{q \le Q} \max\limits_{y \le x} E(y,q) = O\left(Q \cdot x^{1/2} (\log x)^5\right)$.

This shows that average error term is $O(x^{1/2}(\log x)^5)$, as good as RH. This theorem is beyond the scope of this course...